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349. Proposed by JOSEPH A. NYBERG, Student, University of Chicago.

To show that the determinant of the n th order:

$$D_n = \begin{vmatrix} C & -1 & 0 & 0 & 0 & 0 & 0 & . & . \\ -1 & C & -1 & 0 & 0 & 0 & 0 & . & . \\ 0 & -1 & C & -1 & 0 & 0 & 0 & . & . \\ 0 & 0 & -1 & C & -1 & 0 & 0 & . & . \\ 0 & 0 & 0 & -1 & C & -1 & 0 & . & . \\ 0 & 0 & 0 & 0 & -1 & C & -1 & . & . \\ . & . & . & . & . & . & . & . & . \end{vmatrix}$$

has the value: $D_n = C^n + \sum_{r=1}^n (-1)^r \frac{(n-r)(n-r-1) \dots (n-2r+1)}{r!} C^{n-2r}$.

I. Solution by the PROPOSER.

If the determinant is expanded by the elements of the first row or column, we get:

$$D_n = C \cdot D_{n-1} - D_{n-2},$$

where D_{n-1} and D_{n-2} are determinants similar to D_n but having, respectively, one and two fewer rows and columns. Compare

$$(1) \quad D_n + D_{n-2} = C \cdot D_{n-1}$$

with

$$(2) \quad \sin(a+b) + \sin(a-b) = 2 \sin a \cos b.$$

Put $a = n\theta$, $b = \theta$. Then (2) becomes

$$\sin(n+1)\theta + \sin(n-1)\theta = 2 \sin n\theta \cos \theta.$$

Now put

$$(3) \quad C = 2 \cos \theta \text{ and } D_n = c \sin(n+1)\theta,$$

where c is independent of n . To determine its value we note that, for $n=1$,

$$D_1 = C = 2 \cos \theta \equiv \frac{\sin 2\theta}{\sin \theta}.$$

$$(4) \quad \therefore c = \csc \theta.$$

As a result of (3) and (4) we can write:

$$D_n = \frac{\sin(n+1)\theta}{\sin \theta}.$$

But, from Chrystal's *Algebra*, Vol. 2, p. 252, we have:

$$(5) \frac{\sin(n+1)\theta}{\sin \theta} = (2\cos \theta)^n + \sum_{r=1}^n (-1)^r \frac{(n-r)(n-r-1)\dots(n-2r+1)}{r!} (2\cos \theta)^{n-2r}$$

Using equations (5) and (3) the desired formula is obtained.

A second solution, though quite long, and therefore will only be sketched here, may be given as follows: We have

$$(1) \begin{aligned} D_i &= C^i + a_{1,i} C^{i-1} + a_{2,i} C^{i-2} + a_{3,i} C^{i-3} + \dots \\ D_{i+1} &= C^{i+1} + a_{1,i+1} C^i + a_{2,i+1} C^{i-1} + a_{3,i+1} C^{i-2} + \dots \\ D_{i+2} &= C^{i+2} + a_{1,i+2} C^{i+1} + a_{2,i+2} C^i + a_{3,i+2} C^{i-1} + \dots \end{aligned}$$

But since $D_{i+2} = C.D_{i+1} - D_i$, we have

$$(2) \quad D_{i+2} = C^{i+2} + a_{1,i+1} C^{i+1} + (a_{2,i+1} - 1) C^i + (a_{3,i+1} - a_{1,i}) C^{i-1} + \dots$$

Comparing (2) with the last of (3), it can be shown that $a_{jp} = 0$ when j is odd. Consequently the powers of C diminish by 2.

By studying the successive coefficients of powers of C we can show by induction that the coefficient of $C^{(n-2r)}$ is

$$(-1)^r \frac{(n-r)(n-r-1)\dots(n-2r+1)}{r!},$$

which then gives the formula as originally stated.

II. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology.

Expanding in terms of the first column, we have the following relation connecting three determinants of the kind here considered whose orders are n , $n-1$, $n-2$:

$$D_n = CD_{n-1} - D_{n-2}.$$

Forming some of the successive values of D we find:

$$\begin{aligned} D_1 &= C, \\ D_2 &= C^2 - 1, \\ D_3 &= C^3 - 2C, \\ D_4 &= C^4 - 3C^2 + 1, \end{aligned}$$

$$\begin{aligned}
D_5 &= C^5 - 4C^3 + 3C, \\
D_6 &= C^6 - 5C^4 + 6C^2 - 1, \\
D_7 &= C^7 - 6C^5 + 10C^3 - 4C, \\
D_8 &= C^8 - 7C^6 + 15C^4 - 10C^2 + 1.
\end{aligned}$$

It is clear that starting with D_n and reading the terms diagonally we have the expansion of $(C-1)^n$. For instance, starting with D_4 we have $C^4 - 4C^3 + 6C^2 - 4C + 1$. Hence, reading horizontally, the 1, 2, 3, 4, etc., terms of D_n will be the 1, 2, 3, 4, etc., terms in the expansions of $(C-1)^n$, $(C-1)^{n-1}$, $(C-1)^{n-2}$, $(C-1)^{n-3}$, etc., respectively. The r th term will be the r th term of the expansion of $(C-1)^{n-r+1}$. Hence

$$D_n = C^n - (n-1)C^{n-2} + \frac{(n-2)(n-3)}{2!}C^{n-4} - \frac{(n-3)(n-4)(n-5)}{3!}C^{n-6} + \dots$$

$$\text{or } D_n = C^n + \sum_{r=1}^n (-1)^r \frac{(n-r)(n-r-1)\dots(n-2r+1)}{r!} C^{n-2r}.$$

A very neat solution of this problem was also received from a contributor who failed to sign his name to the solution. Will contributors please note that we wish them to put their names to the solutions and to observe the order of the printed solutions, viz., put name at beginning of solution rather than at the end?

GEOMETRY.

369. Proposed by W. J. GREENSTREET, A. M., Editor, *Mathematical Gazette*, Stroud, England.

Prove by inversion that if two circles cut at a given angle, touch each a given circle, and pass each through the same fixed point, then shall the envelope of the points of contact be a conic.

Discussion by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Since all of the contact points lie on the fixed circles it seems probable that the desired locus is that of the second point of intersection of the variable circles. In Fig. 1, let S_1 and S_2 be the fixed circles, T_1 and T_2 the variable circles passing through the fixed point Q at the constant angle θ .

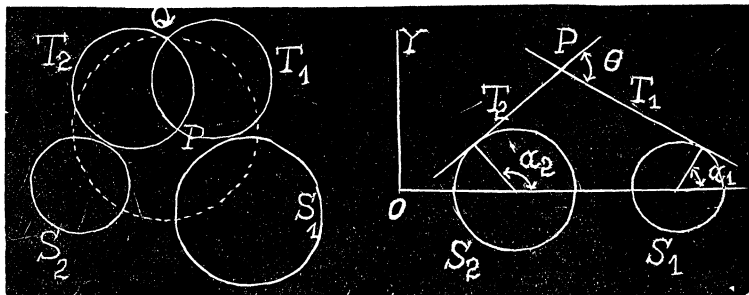


Fig. 1.

Fig. 2.